

Development of dispersion curves for two-layered cylinders using laser ultrasonics

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In this paper, laser-ultrasonic techniques are employed to develop a quantitative understanding of the underlying principles of the propagation of guided circumferential waves in two-layered cylindrical components. The high-fidelity, broad-bandwidth, point source/receiver and noncontact nature of these optical techniques are critical elements to the success of this work. The experimental procedure consists of measuring a series of transient, circumferentially propagating waves in a cylindrical waveguide and then operating on these transient waveforms with signal-processing techniques to develop the dispersion relationship for that waveguide; this procedure extracts the steady-state behavior from a series of transient measurements. These dispersion curves are compared to theoretical values. There is good agreement between the experimental and theoretical results, thus demonstrating the accuracy and effectiveness of using laser-ultrasonic techniques to study the propagation of guided circumferential waves. © 1999 Acoustical Society of America. [S0001-4966(99)05707-0]

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INTRODUCTION

There exists a widespread need for methodologies that can nondestructively evaluate cylindrical components such as a helicopter rotor hub. Of particular interest is a geometry which consists of a hollow outer cylinder with either a solid or hollow inner-cylindrical shaft. With this specific two-layered geometry, fatigue cracks usually develop at the interface (at the outer surface of the inner shaft and the inner surface of the outer hollow cylinder) and grow in the radial direction. Unfortunately, conventional ultrasonic methodologies (such as pulse-echo) are difficult to implement in these components, mainly because of accessibility issues. A promising new methodology uses guided circumferential waves to examine cylindrical components. The primary advantage with using these guided waves is that they are capable of interrogating the entire component, including inaccessible regions of a complex structure. On the other hand, the main difficulty with the application of guided waves for the evaluation of cylindrical components is the inherent complexity of the waveforms, making interpretation difficult.

This research employs a laser-ultrasonic technique to develop a quantitative understanding of the underlying principles of the propagation of guided circumferential waves in two-layered cylindrical components. The high-fidelity,

broad-bandwidth, and noncontact nature of the optical technique are critical for the success of this work. In addition, laser ultrasonics allows for measurements with a point source and a point receiver, thus enabling spatial sampling techniques such as the two-dimensional Fourier transformation (2D-FFT). By using these state-of-the-art laser-ultrasonic methodologies, it is possible to experimentally measure transient waves in a variety of cylindrical specimens without any of the frequency biases present in, for example, piezoelectric transducers.

It is important to note that the proposed experimental procedure makes measurements of *transient* waveforms in cylindrical waveguides. While these transient waveforms provide valuable information about the propagation of guided circumferential waves, they are not in themselves sufficient to quantitatively understand the propagation of guided waves in cylindrical waveguides; this comprehension requires an understanding of the *steady-state (time harmonic)* behavior of wave propagation in cylindrical waveguides. Steady-state behavior of guided waves is best interpreted in terms of dispersion curves, which present the relationship between frequency and phase velocity (or wave number) for each of the infinite number of modes possible in a particular waveguide. In order to experimentally model (and understand) steady-state behavior, this research consists of measuring a series of transient waveforms and operating on them with signal-processing techniques to infer the dispersion

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curves of the waveguide. There are two signal-processing techniques used in this research: the 2D-FFT and the Prony method. Each of these signal-processing techniques requires multiple transient waveforms (generated with exactly the same source), each having a different propagation distance, separated by an equally spaced increment. In this study, these multiple, equally spaced measurements are made with a repeatable optical source, the pulse of an Nd-YAG laser. By using a fiber-optic delivery system to launch this laser pulse, it is possible to provide a repeatable source at equally spaced increments.

There has been extensive research into the propagation of guided waves in flat, layered components (Chimenti¹), but there is much less information on the propagation of guided waves in cylindrical, layered components. Valle *et al.*² recently developed an analytical model to obtain the dispersion relationship for guided waves that propagate in the circumferential direction of a two-layered cylinder; one motivation for the current research is to obtain an experimental comparison that endorses the theoretical results developed in Ref. 2. Hutchins and Lundgren³ and Schumacher *et al.*⁴ used laser-ultrasonic techniques to study the propagation of transient Lamb waves in (flat) layered materials. Kawald *et al.*⁵ examined circumferential surface waves in a thin layer bonded to a solid cylinder, while Liu and Qu⁶ studied guided circumferential waves in a circular structure. Note that solutions exist for propagation in the axial direction of a hollow cylinder (e.g., Gazis⁷), but this paper addresses propagation in the circumferential direction.

The objective of the present study is to develop and evaluate laser techniques for the noncontact generation and detection of transient, broadband ultrasonic signals in layered cylindrical structures. The experimental procedure measures ultrasonic waveforms over an array of closely spaced points, and signal-processing techniques are then applied to this array data to determine dispersion curves over a wide frequency range (200 kHz to 10 MHz). In future work, this system will be used to study the effects of imperfections (such as fatigue cracks) on the propagation of circumferential waves in cylindrical structures.

I. EXPERIMENTAL PROCEDURE

Guided waves are generated with the pulse of a Q-switched Nd:YAG laser that is launched into an optical fiber. Laser generation of ultrasound in a metal creates a repeatable, broadband ultrasonic waveform; see Scruby and Drain⁸ for details. The Nd:YAG laser (1064 nm) used in this study emits a 450-mJ, 4–6-ns pulse with a spatially Gaussian profile. The beam is attenuated and focused before it is launched into the optical fiber. The other end of the optical fiber is mounted on a rotation stage with a fine micrometer adjustment with a graduation of 0.013283° that allows for a minimum circumferential (arc) increment of $14.721\ \mu\text{m}$ at a radius of 63.5 mm (the outer radius of each specimen being interrogated). The specimen is mounted in the center of the rotation stage to ensure that the fiber end is kept at a constant distance (about 1 mm) from a specimen's surface throughout each experiment (see Fig. 1). This setup provides a laser source that generates exactly the same ultrasonic signal, at

multiple, equally spaced locations, throughout each experiment. Note that this setup does not guarantee that the laser source is exactly the same (spot size of approximately 1 mm) for all the specimens (or for a specimen that is removed and reinstalled), just that the laser source remains constant as the fiber is rotated to different source locations on the same specimen.

Laser detection of these guided waves is accomplished with a dual-probe, heterodyne interferometer that is a modified version of the instrument described in detail in Bruttomesso *et al.*⁹ This optical device uses the Doppler shift to simultaneously measure out-of-plane surface velocity (particle velocity) at two points on the specimen's surface. The interferometer works by measuring frequency changes in the light reflected off the specimen surface. In a heterodyne interferometer, a frequency shift is initially imposed (using an acousto-optic modulator) to create a reference and a probe beam. The probe beam is reflected off the specimen surface and is recombined with the reference beam at a photodiode. This creates a beat frequency equal to the initially imposed frequency shift. Frequency shifts in the light reflected from the specimen surface result in proportional shifts in the beat frequency. As a result, the beat-frequency signal acts as a carrier that is demodulated in real time (with an FM discriminator) to obtain the surface velocity. The interferometer makes high-fidelity, absolute measurements of surface velocity (particle velocity) over a bandwidth of 200 kHz to 10 MHz.

It is important to note that the proposed experimental procedure requires only a single ultrasonic receiver (probe). However, this work uses the second receiver (probe) as a redundant check on the consistency of the optical source. An additional use of the signals measured with the second probe is for the development of dispersion relationships with the two-point phase method.¹⁰ Another study¹¹ showed that the two-point phase method is not as robust and quantitative in determining dispersion relationships for these cylindrical specimens.

Note that all of the waves presented are low-pass filtered at 10 MHz. In addition, in order to increase the signal-to-noise ratio (SNR), each waveform presented represents a collection of averages, sometimes as many as 100. This signal-averaging procedure works because noise is random, while the "real" signal is repeatable; SNR is improved by the square root of N , where N is the number of averages.

Three different aluminum specimens are examined in this study. The first specimen is a hollow cylinder with an outer diameter of 127 mm and a wall thickness of 4 mm. The second specimen consists of the same hollow aluminum cylinder with an inner aluminum (solid) shaft (119-mm diameter). A thin layer of 10W-40 motor oil provides a slip-boundary condition at the interface between the inner shaft and the outer cylinder. The third specimen combines the original outer, hollow-aluminum cylinder with an inner hollow-aluminum cylinder (119-mm outer diameter and a wall thickness of 4 mm) and the same slip interface. Note that all three specimens have the same outer cylinder, and that the surface of this cylinder is sanded and polished with a polishing paste. This surface preparation enables true non-

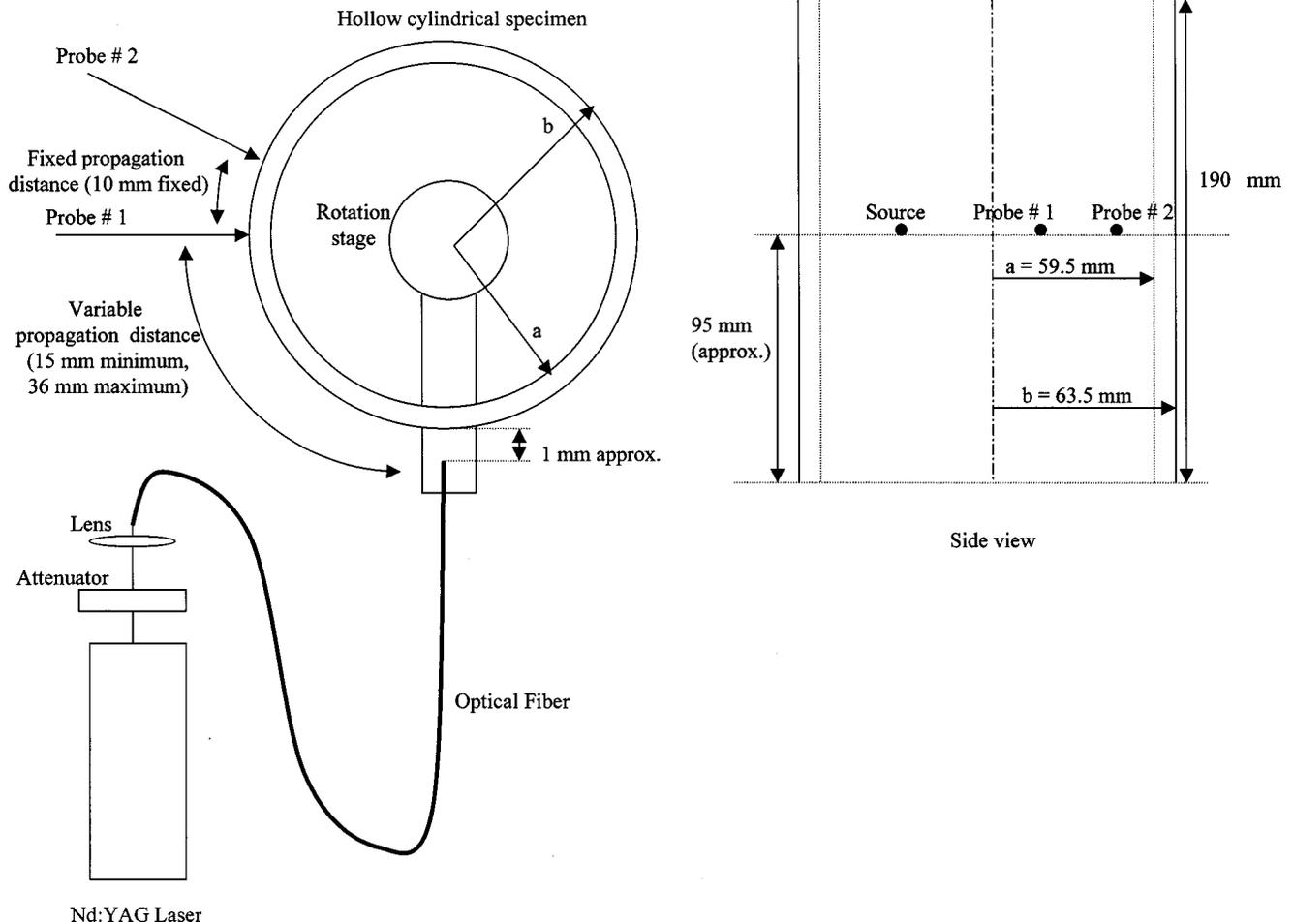


FIG. 1. Schematic of optical-fiber delivery system and hollow specimen.

contact detection (there are no artificial surface treatments such as reflective tape) and provides a consistent surface for guided wave propagation. The length of each specimen is 190 mm and all measurements are made in the vicinity of the center line (see Fig. 1). As a result, these specimens are treated as infinitely long cylinders—reflections from the ends arrive well after the signal of interest.

II. EXPERIMENTALLY MEASURED TRANSIENT WAVEFORMS

While the receiver is kept in a fixed position, the source is placed at 49 equally spaced locations. Incremental (circumferential) distances of 0.4 mm separate these multiple source locations. Note that the closest source-to-receiver (propagation) distance is 15 mm, while the farthest propagation distance is 36 mm. This results in the (generation and) detection of 49 waveforms, each with a different propagation distance and each generated with exactly the same laser source.

Figure 2 shows three typical transient waveforms with the same propagation distance (15 mm), but measured in the three different specimens (the hollow cylinder, the hollow cylinder with a solid inner shaft, and the hollow cylinder with a hollow inner cylinder). Note that the amplitudes of all three waveforms are normalized to enable a representative

comparison and consider the outstanding SNR exhibited in these waveforms; high SNR is critical for the success of the next step—development of dispersion curves. A qualitative analysis of these three waveforms shows identical *arrival times* for both the beginning of the signal (the first nonzero disturbance at $2 \mu\text{s}$) and the Rayleigh wave (mode), but significant differences between the shapes of each signal. The identical arrival times of the Rayleigh waves is not surprising, since they all propagate along the same outer cylinder. The variances in shape are due to differences between the three specimens (the presence of the inner shaft and inner cylinder), as well as the presence of the slip boundary (interface). Unfortunately, besides matching arrival times and making generic comments about changes in shape, no other analysis is possible. As a result, it is impossible to quantitatively interpret these transient waveforms. However, it is possible to develop dispersion curves and then quantitatively interpret these results.

III. DEVELOPMENT OF DISPERSION CURVES

This research uses two different signal-processing techniques to develop dispersion curves from each set of (experimentally measured) equally spaced, transient waveforms: the 2D-FFT and the Prony method. Implementation of the 2D-FFT is fairly straightforward—perform a temporal Fourier

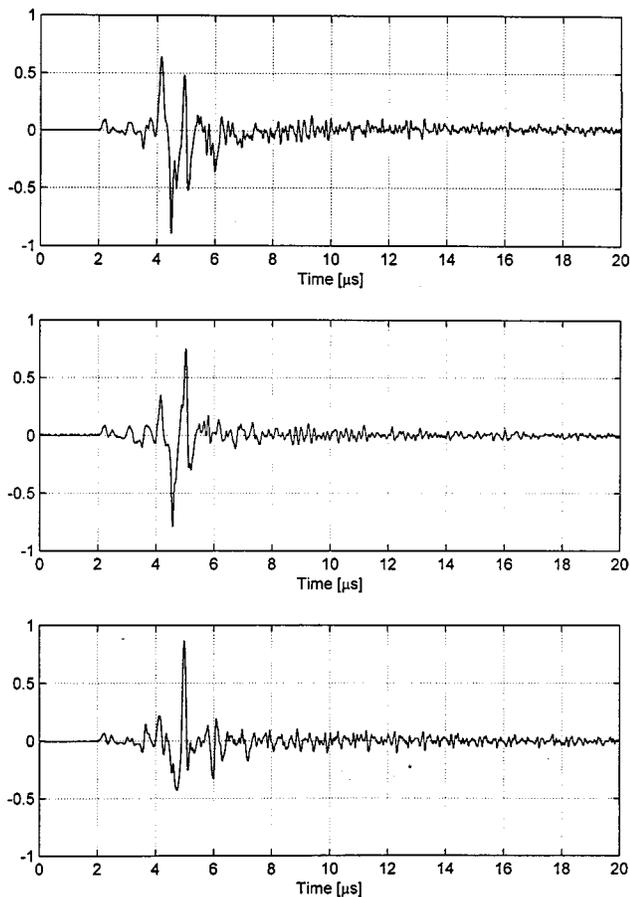


FIG. 2. Comparison of three typical guided circumferential waves with the same propagation distance (15 mm), but measured in the hollow cylinder, the hollow outer cylinder with a solid inner shaft, and the hollow outer cylinder with a hollow inner cylinder (top, middle, and bottom, respectively).

transform (going from the time to the frequency domain) followed by a spatial Fourier transform (going from the spatial to the wave number domain). The resulting frequency (f) versus wave number (k) spectrum shows a series of peaks that represents individual modes (see Alleyne and Cawley¹² or Moser *et al.*¹³). In order to be effective, the 2D-FFT requires a large number of broadband, transient signals measured with a small (incremental) spatial sampling distance. Note that the 2D-FFT does not require (or use) any prior knowledge concerning the waveguide being modeled.

In contrast, the Prony method requires knowledge of the propagating modes in a waveguide. If the number of modes at a specific frequency is known *a priori*, the Prony method fits that number of modes to the transient data (see Glandier *et al.*¹⁴). The objective of the Prony method is to find a set of p exponentials of arbitrary complex amplitude (magnitude and phase) and complex wave number (wave number and attenuation) that best approximates the real, spatial data. The Prony algorithm is particularly appropriate when the signal is indeed a sum of exponentials, and when the order of the model (i.e., the number of exponentials) is known (or estimated) *a priori*. The Prony method (like the 2D-FFT) requires waveforms from equispaced sources. Unfortunately, noise in a signal has more of a negative effect on the Prony method than on the 2D-FFT. Although the Prony method can

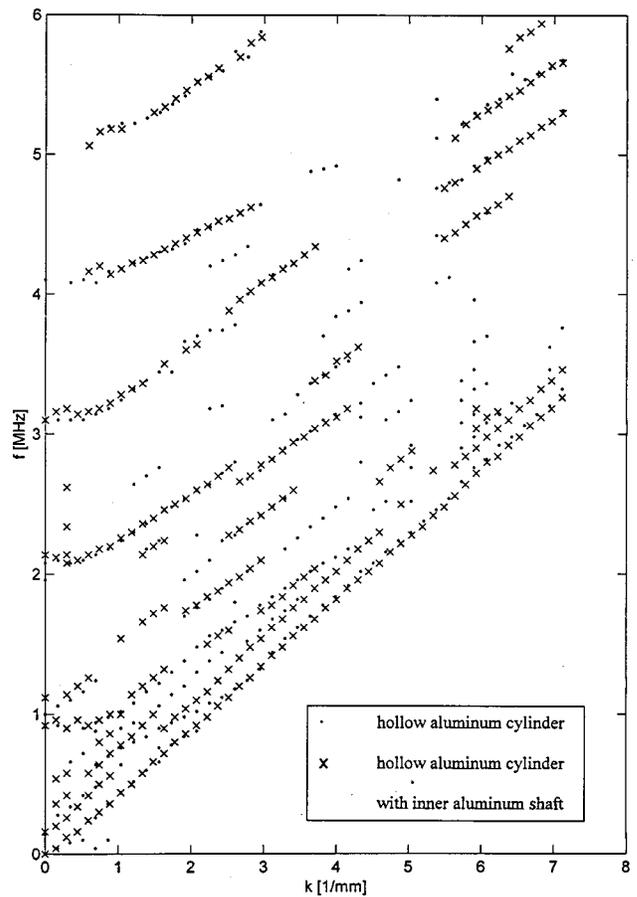


FIG. 3. Comparison of k - f spectrum for hollow cylinder with spectrum of hollow outer cylinder with solid inner shaft (2D-FFT).

theoretically identify a number of modes that is equal to half of the number of spatially sampled signals, the presence of noise will severely reduce this limit.¹⁴ The procedure described in Ref. 14 and used in the present study is a basic Prony method that is relatively simple to apply. Recently, an extended Prony method¹⁵ has been developed which includes a systematic procedure for discriminating against spurious peaks in the wave number spectrum due to noise.

The Prony method is implemented with a two-step process. First, perform a spatial Fourier transform (from the spatial to the wave number domain) and second, determine the amplitudes for these wave numbers by implementing a linear, least-squares fit to a model of the data (see Refs. 11 or 14 for details).

IV. EXPERIMENTALLY CALCULATED DISPERSION CURVES

The 2D-FFT is used to calculate the dispersion curves for each of the specimens; the results indicate that certain k - f combinations have significant amplitudes (peaks), and these combinations are solutions to the specimens' dispersion relationships (the specimens' individual modes). The local maxima in the vicinity of each peak for both the hollow cylinder and the cylinder with solid shaft are plotted in Fig. 3, while Fig. 4 is a comparison of the k - f peaks for the hollow cylinder and the cylinder with an inner hollow cylinder.

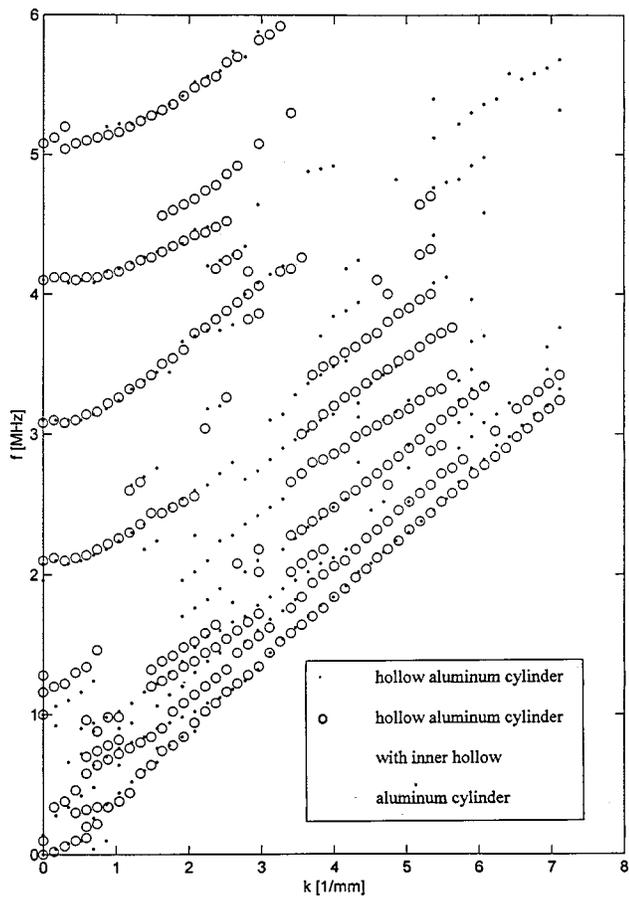


FIG. 4. Comparison of k - f spectrum for hollow cylinder with spectrum of hollow outer cylinder with hollow inner cylinder (2D-FFT).

These figures show that all three specimens have the same first mode—the Rayleigh mode; the Rayleigh mode is nearly a straight line, so it is (effectively) nondispersive. There is a series of higher modes that have clear and definitive cutoff frequencies, indicating that these higher modes exist only above a specific lower-frequency value. In addition, dispersion is clearly visible in these higher modes; they follow a curve in the beginning, and turn into straight lines at higher wave numbers (or frequencies). Overall, the results from the 2D-FFT provide excellent definition and clarity of the individual modes that are present in each specimen.

Next, the Prony method is used to operate on the same set of transient waveforms. For example, Fig. 5 shows the k - f spectra for the hollow cylinder and the hollow cylinder with the inner shaft. These spectra are very similar to those developed with the 2D-FFT, each exhibiting the same general trends. Note that the Prony method allows for more “control” over the selection of each mode than is possible with the 2D-FFT. For example, the Prony method involves an iterative process, with a large amount of user flexibility to eliminate (or ignore) certain spectrum points. As a result, it is possible to exert a deliberate bias and create artificially good results; this level of control is not possible with the 2D-FFT. However, an advantage of the Prony method is that the dispersion curves are calculated on a frequency-by-frequency basis, which enables a high degree of refinement in frequency ranges of interest. As a result, the Prony method

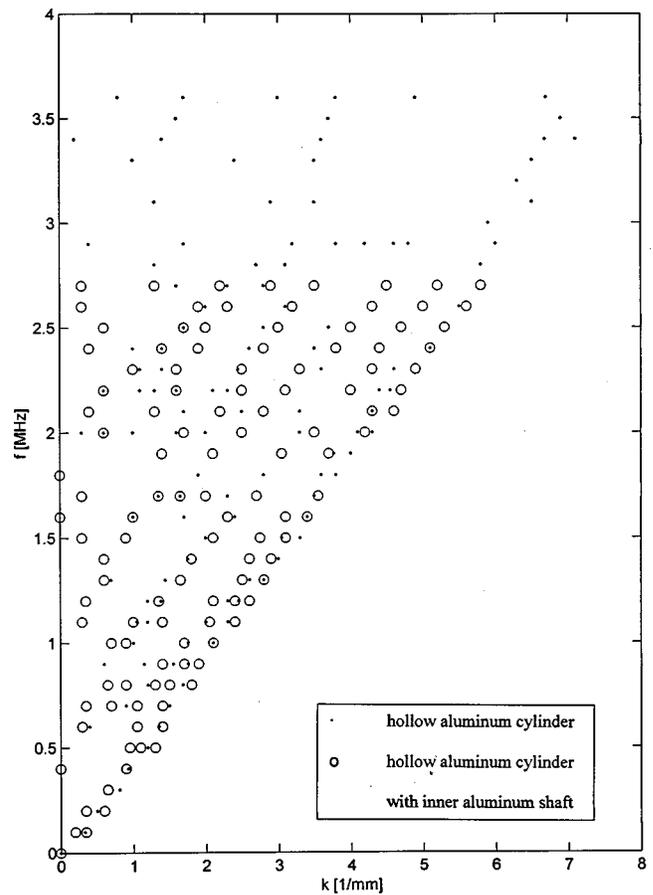


FIG. 5. Comparison of k - f spectrum for hollow cylinder with spectrum of hollow outer cylinder with solid inner shaft (Prony method).

is capable of separating closely spaced modes.

The fact that all three specimens have the same first (Rayleigh) mode is not surprising since they all correspond to a surface wave that propagates on the same outer surface. The second modes show differences between the hollow cylinder and the other two specimens. For example, in the hollow cylinder, the first and second modes approach each other at around 1 MHz; they are effectively inseparable (above 1 MHz) in the 2D-FFT results. In contrast, the second modes in each of the other two specimens remain well separated from their respective first modes, and continue on a parallel slope equal to the Rayleigh wave-phase velocity. Valle *et al.*² theoretically determined the effects of an inner shaft on the first five modes and showed that for a layered cylinder at higher frequencies, the second mode is actually an additional Rayleigh wave that propagates along the free-sliding interface. Next, consider the differences between each specimen in the vicinity of 1 MHz. The hollow cylinder has two modes with a cutoff frequency of 1 MHz, while the other two specimens each show one mode starting below 1 MHz and one mode starting above 1 MHz. However, there is much overlap and not many significant differences when comparing the higher modes (those with cutoff frequencies above 2 MHz) for all three specimens. The similarities for these higher modes show that the inner cylinder (or shaft) has little effect on the experimentally measured dispersion

curves above a certain frequency (2 MHz for this configuration).

It is important to note that all of these dispersion relationships are calculated from transient waveforms that are measured with a receiver (the heterodyne interferometer) that only measures the out-of-plane component of surface velocity. As a result, there is a certain bias in this frequency spectrum since it only contains modes that excite significant out-of-(the radial)-plane (or flexural) motion. This experimental procedure does not identify modes that primarily excite in-plane (longitudinal) motion. Consequently, the frequency spectra presented in Figs. 3–5 are not necessarily the complete spectra—certain in-plane portions of the modes are missing. However, there are no “pure” flexural or longitudinal modes for circumferential waves that propagate in a cylinder,² so Figs. 3–5 provide a very good representation of the dispersion curves for these cylinders.

V. COMPARISON WITH THEORY

The accuracy and validity of these experimentally obtained dispersion curves is determined by comparing them with the theoretical results developed in Ref. 2. Valle *et al.*² uses two-dimensional linear elasticity to develop the dispersion relationship for time harmonic, circumferential waves in (infinitely long) two-layered cylinders. This work uses potential functions that represent guided waves that propagate in the circumferential direction, which is a natural extension of Lamb waves in a flat plate. The boundary conditions are as follows: (1) zero traction at the outer surface; (2) slip-boundary condition at the interface (continuous radial tractions and displacements, plus zero shear stress at the interface); and (3) zero traction at the inner surface (when the second cylinder is hollow). The potentials are written in terms of Bessel functions of the first and second kind. The resulting system of six homogeneous equations is written in matrix form, and the condition that the determinant of this matrix must vanish yields a characteristic equation that is solved numerically. The numerical solution involves selecting a specific (nondimensional) wave number, and using a bisection root-finding scheme to solve the characteristic equation for the corresponding (nondimensional) frequency. Implementation of the numerical solution requires extreme care, since the determinant experiences rapid changes in the vicinity of its roots.²

For example, Fig. 6 compares the first eight theoretical modes of the hollow cylinder with those obtained experimentally (with the 2D-FFT). As a more detailed comparison between these theoretical and experimental data, Fig. 7 isolates the results for the second through fifth modes only. Next, Fig. 8 compares the first three theoretical modes (with the experimental results from the 2D-FFT) for the hollow cylinder with the solid inner shaft. There is good agreement between the spectra of both these specimens; these figures clearly demonstrate the accuracy of the experimentally measured dispersion curves. The main disparities occur in the second mode in the hollow cylinder (where the experimental data is inseparable from the first mode above 1 MHz) and the lack of separation between the second and third modes in the

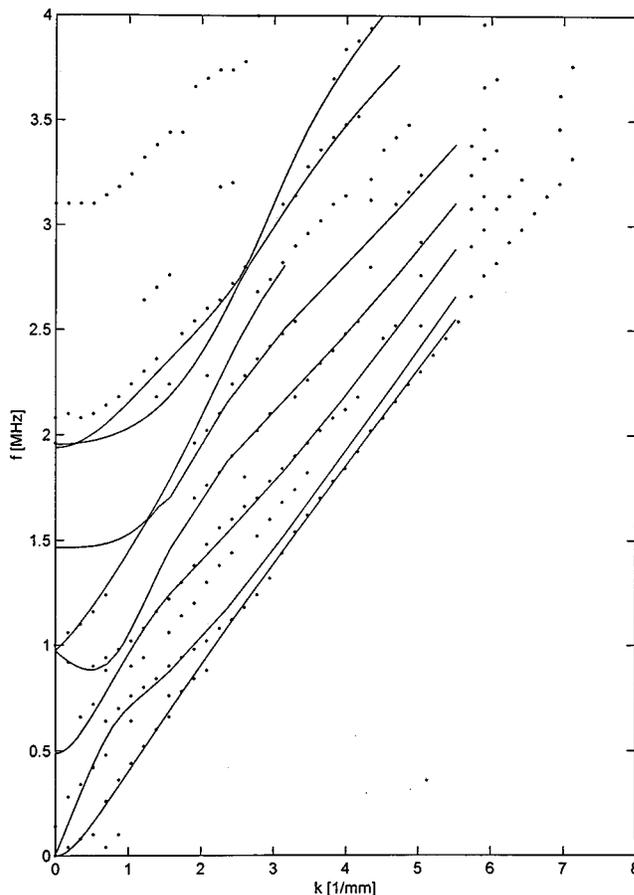


FIG. 6. Comparison of experimental k - f spectrum (2D-FFT) and first eight theoretical modes for hollow cylinder.

hollow cylinder with solid shaft. These disparities are ascribed to difficulties in using the 2D-FFT to extract closely spaced modes.

It should be noted that a theoretical flat-plate model would not completely capture the physics of the high-frequency circumferential waves that propagate in the relatively thin cylinder of this study. Liu and Qu⁶ showed that

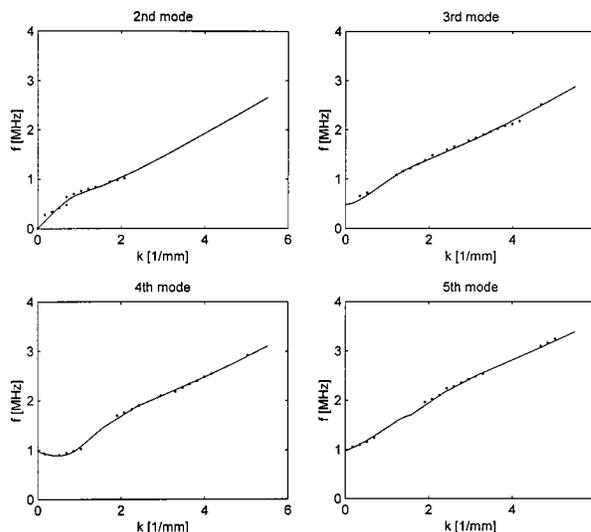


FIG. 7. Detailed comparison of experimental k - f spectrum (2D-FFT) and second through fifth modes (theoretical) for hollow cylinder.

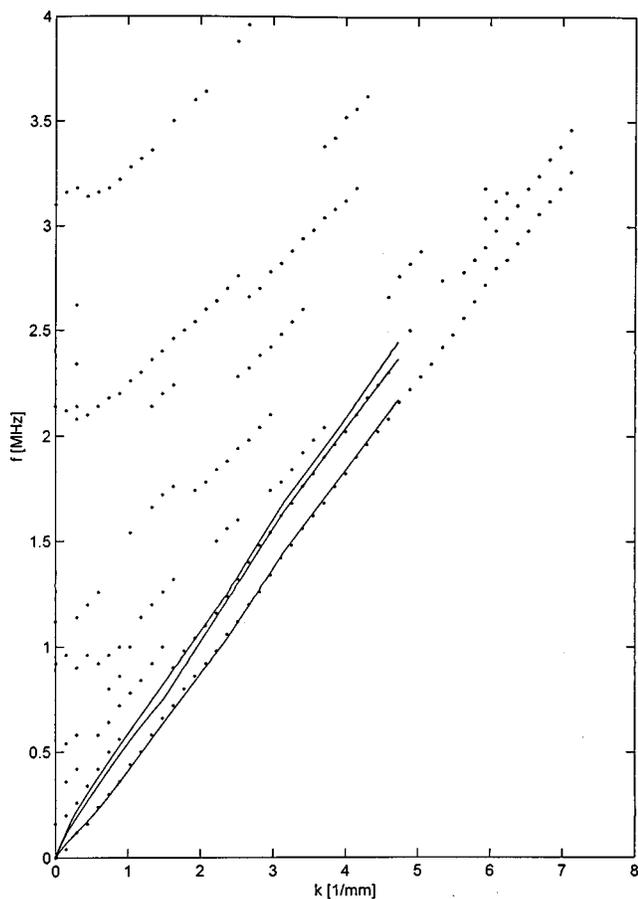


FIG. 8. Comparison of experimental k - f spectrum (2D-FFT) and first three theoretical modes for hollow outer cylinder with solid inner shaft.

the propagating wave fields in a thin shell and a flat plate are quite different, even though their dispersion curves might be similar.

VI. CONCLUSION

This paper demonstrates the effectiveness of using laser-ultrasonic techniques to study the propagation of guided waves in layered cylinders. By using this innovative technique, it is possible to experimentally measure transient waves in a variety of cylindrical specimens and to study the influence of a number of geometric features on these guided waves. It is critical to note that these experimental measurements are possible only because of the high-fidelity, unbiased, broadband, point source/point receiver and noncontact nature of laser ultrasonics.

One objective fulfilled by this research is to increase the understanding of the underlying principles of the propagation of guided circumferential waves by making benchmark, high-fidelity measurements of transient waves in two-layered cylinders. Another specific accomplishment of this research is a demonstration that it is possible to use a fiber-optic delivery system, coupled with a heterodyne interferometer, to generate and detect guided waves from multiple, equally spaced source locations, with an excellent SNR.

As a direct result of the high SNR of these experimentally measured transient waveforms, it is possible to use

signal-processing techniques to calculate dispersion curves, and thus determine the steady-state behavior of cylindrical waveguides. This work achieves success with two different signal-processing techniques: the 2D-FFT and the Prony method. Since each of these signal-processing techniques can operate on exactly the same set of equally spaced transient waveforms, it is possible to compare the accuracy and robustness of each technique. This research shows that while both methods are extremely effective, the 2D-FFT is more suitable in identifying well-spaced modes, while the Prony method is better at separating a large number of closely spaced modes.

A final motivation for the current research is to experimentally validate the theoretical model developed in Ref. 2. A comparison of theoretical and experimental results demonstrates the fidelity and accuracy of the experimental measurements made in this research, and there is good agreement.

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